Scaling exponents for turbulence and other random processes and their relationships with multifractal structure

Samuel I. Vainshtein
Department of Astronomy and Astrophysics, University of Chicago, Chicago, Illinois 60637

K. R. Sreenivasan
Mason Laboratory, Yale University, New Haven, Connecticut 06520

Raymond T. Pierrehumbert
Department of Geophysical Sciences, University of Chicago, Chicago, Illinois 60637

Vinay Kashyap
Department of Astronomy and Astrophysics, University of Chicago, Chicago, Illinois 60637

Anurag Juneja
Mason Laboratory, Yale University, New Haven, Connecticut 06520
(Received 13 May 1994)

In the recent literature on high-Reynolds-number turbulence, several different types of scaling exponents—such as multifractal exponents for velocity increments, for energy and scalar dissipation, for the square of the local vorticity, and so forth—have been introduced. More recently, a new exponent called the cancellation exponent has been introduced for characterizing rapidly oscillating quantities. Not all of these exponents are independent; some of them are simply related to more familiar scaling for velocity and temperature structure functions either exactly or through plausible hypotheses familiar for turbulence. A primary purpose of this paper is to establish the interrelationships among the various exponents. In doing so, we obtain several additional relations. Much of the paper is relevant to general stochastic processes, although the discussion is heavily influenced by the turbulent context. We first examine the case of one-dimensional random processes and subsequently consider two- and three-dimensional processes. Special consideration is given to characteristic values appropriate to the geometry of turbulence, as well as the lifetimes of eddies of various scales. Finally, we discuss some properties of the tails of the probability density function to which the scaling properties of high-order structure functions are related and discuss the implications of multifractality on their structure.

PACS number(s): 05.45.+b, 47.27.−i, 52.35.−g

I. INTRODUCTION

Consider a three-dimensional random vector field $\mathbf{v}(\mathbf{x})$ with its average value $\langle \mathbf{v} \rangle = 0$. To describe certain statistical aspects of this field, Kolmogorov [1] introduced the so-called structure functions as

$$\langle (\Delta \mathbf{v}_r)^n \rangle = \langle [\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})]^n \rangle \ , \qquad (1.1)$$

where $\mathbf{r}$ is the separation vector and n is an integer. We can expect some scaling laws for these functions,

$$\langle (\Delta \mathbf{v}_r)^n \rangle = C_n r^{\xi_n} \ . \qquad (1.2)$$

Obviously, for $n=1$, $C_1=0$.

Note that the structure functions (1.1) can be defined only for integer values of n. In analogy with (1.1) one can define the so-called generalized structure functions for any real value of q as

$$\langle |\Delta \mathbf{v}_r|^q \rangle = \langle |\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})|^q \rangle \ . \qquad (1.3)$$

For these generalized structure functions, one can define the scaling exponents

$$\langle |\Delta \mathbf{v}_r|^q \rangle \sim r^{\zeta_q} \ . \qquad (1.4)$$

Clearly, when q is an even integer, the scaling exponents defined via (1.2) and (1.4) are identical.

Let the random vector field under consideration represent turbulent velocity field at high Reynolds numbers. Kolmogorov [1] showed, under certain hypothesis which are now very familiar, that

$$\xi_n = \frac{n}{3} \qquad (1.5)$$

in the so-called inertial range (i.e., the range of scales which are large compared to the viscous scale and small compared to the large scale of turbulence). For the special case $n=3$, it is well known [2] that the relation (1.5) is exact when the Reynolds number $\mathrm{Re} \to \infty$ and that

$$\langle (\Delta \mathbf{v}_r)^3 \rangle = -\tfrac{4}{5} \langle \epsilon \rangle \mathbf{r} \ , \qquad (1.6)$$

where $\langle \epsilon \rangle$ is the average rate of dissipation per unit mass. On the other hand, numerous experiments [3] have shown that ξ_n are different from $n/3$ (except for $n=3$). These departures are thought to arise from the intermittency of the inertial range turbulence. Many models [4] incorporating intermittency have been proposed.

Over the past dozen or so years, other scaling exponents have also been introduced. For a D-dimensional random field ω, the "critical dimensions" D_q for $q>1$ have been introduced by Mandelbrot [5]. Namely, along a cut of dimension D through a measure, one has $\langle|\omega(r)|^q\rangle=\infty$ when $q>q_{\text{crit}}(D)$. The function inverse to $q_{\text{crit}}(D)$ is D_q, which have since become known as generalized dimensions. This can be seen from the formula (see also [6])

$$\langle|\omega(r)|^q\rangle=C_q\langle|\omega|\rangle^q\left[\frac{r}{l}\right]^{-(D-D_q)(q-1)}, \tag{1.7}$$

where the local mean value has been defined as

$$|\omega|_i=\frac{1}{r^D}\int_{C_i}|\omega|dx \tag{1.8}$$

and the brackets denote the global average

$$\langle|\omega(r)|^q\rangle=\frac{\sum_i|\omega|_i^q}{N}. \tag{1.9}$$

Here $N=1/r^D$ is the number of boxes of size r into which the volume $V=1$ has been divided. The generalized dimensions D_q are based on the probability measure

$$\mu_\omega(C_i)=\frac{\int_{C_i}|\omega|dx}{\int_V|\omega|dx}. \tag{1.10}$$

Coefficients C_q are of order of unity and, in principle, may be substantially smaller than one for high values of q.

Much work [7] has been done towards the experimental determination of the generalized dimension D_q when $|\omega|$ stands for the absolute values of the velocity increments, scalar and energy dissipation, as well as the absolute value of local vorticity in turbulent flows. Note that the generalized dimensions are simply related [8] by Legendre transform to the so-called $f(\alpha)$ curve, which represents an infinity of Hausdorff dimensions for each iso-α set of singularity strength α.

Note that the exponents defined so far do not explicitly focus on the oscillatory character of the process ω. In an effort to characterize the rapid oscillatory character of turbulence quantities such as velocity derivatives, vorticity, and magnetic flux, a new quantity called the cancellation exponent has been defined [9]. This exponent κ is based on the sign-singular measure

$$\mu_s(C_i)=\frac{\int_{C_i}\omega\,dx}{|\int_V\omega\,dx|}. \tag{1.11}$$

With

$$\chi_s(r)=\sum_i\frac{|\int_{C_i}\omega\,dx|}{|\int_V\omega\,dx|}, \tag{1.12}$$

one defines κ as

$$\chi_s(r)=\left[\frac{r}{l}\right]^{-\kappa}. \tag{1.13}$$

There are thus many exponents characterizing high-Reynolds-number turbulence. Not all these exponents contain independent information however. Therefore it seemed useful to attempt to unify these various exponents. This is the first goal of this paper. In this attempt to clarify the interrelationship of these exponents, several scaling relations have been derived in this paper. Finally, since the scaling exponents of high-order-structure functions are related to the tails of the probability density function (PDF) of the velocity increments, we examine this issue as well.

The rest of this paper is organized as follows. Section II contains a discussion of the various exponents for the special case of a one-dimensional random function of time or a one-dimensional cut of D-dimensional random field ($D>1$). This allows a more complete discussion than is possible for the two- and three-dimensional cases, which form the subject of Sec. III. Though the discussion in these sections is cast in the background of turbulence, it is applicable to general random fields. In Secs. IV and V we discuss the characteristic values and the scaling exponents as specialized to turbulence velocity field. Section VI contains a discussion of the lifetimes of turbulent eddies of various scales. The principal conclusion is that intermittency (effectively multifractality) results in much larger lifetimes than otherwise. Section VII contain a discussion of the PDF of velocity increments, and the paper concludes with a summary and discussion in Sec. VIII.

II. STRUCTURE FUNCTIONS AND GENERALIZED DIMENSIONS FOR THE ONE-DIMENSIONAL CASE

A. General expressions

In this section we consider one-dimensional cuts of a higher-dimensional random field or, say, time-dependent random function. Thus, although we continue to use D as if it is general, it has the value of unity in this section. Denote the process under consideration by ω. If one thinks of ω as "vorticity," one then has

$$\omega=\partial_x v, \tag{2.1}$$

where v represents one-dimensional velocity field and x represents the one-dimensional coordinate.

Let the scaling relation Eq. (1.7) hold within the range $l>r\gtrsim r_*$. We write (1.7) for the inner cutoff scale $r=r_*$ as

$$\begin{aligned}\langle|\omega|^q\rangle&=C_q'\langle|\omega|\rangle^q\left[\frac{r_*}{l}\right]^{-(D-D_q)(q-1)}\\&=\omega_l^q\left[\frac{r_*}{l}\right]^{-[q\kappa+(D-D_q)(q-1)]}.\end{aligned} \tag{2.2}$$

Here $\omega_l=v_l/l$ and $v_l=\langle v^2\rangle^{1/2}$. In the second equality in (2.2), the cancellation exponent κ appears. Therefore, one has

$$\chi(r_*)=\left[\frac{r_*}{l}\right]^{-\kappa}=\frac{\langle|\omega|\rangle}{\omega_l} . \qquad (2.3)$$

Substitution of (2.3) into first equality in (2.2) results in the second equality. The new coefficients C_q' in Eq. (2.2) are again of the order of unity.

For the velocity increment

$$\Delta v(x,r)=[v(x+r)-v(x)] , \qquad (2.4)$$

define the structure functions as

$$\hat{S}_n(r)=\langle\Delta v^n\rangle \qquad (2.5)$$

for integer n and the generalized structure functions as

$$S_q(r)=\langle|\Delta v|^q\rangle \qquad (2.6)$$

for any real q; cf. (1.1) and (1.3). Suppose that the scaling range $l>r\gtrsim r_*$ from Eq. (2.6) coincides with the inertial range, i.e., for the same interval of r, the generalized structure functions scale as

$$S_q(r)=\langle|v(x+r)-v(x)|^q\rangle=C_q''v_l^q\left[\frac{r}{l}\right]^{\zeta_q} . \qquad (2.7)$$

It is clear that $|\Delta v|\to|\omega|r_*$ as $r\to r_*$. Therefore, in the same limit, expression $S_q(r)/r^q$ approaches the left-hand side of (2.2). On the other hand, $S_q(r)/r^q$ approaches ω_l^q when $r\to l$, and therefore the only formula which fits both limits has the form

$$\frac{S_q(r)}{r^q}\sim C_q'\omega_l^q\left[\frac{r}{l}\right]^{-[q\kappa+(D-D_q)(q-1)]} . \qquad (2.8)$$

Thus the exponent for the generalized structure function is

$$\zeta_q=(1-\kappa)q-(D-D_q)(q-1) . \qquad (2.9)$$

It is reasonable to relate $S_q(r)/r^q$ to the "correlation functions" of the field ω, defined as

$$\frac{S_q(r)}{r^q}\sim\langle|\omega(x+r)\omega(x)|^{q/2}\mathrm{sgn}\{\,|\,|\,\}\rangle , \qquad (2.10)$$

where

$$\mathrm{sgn}\{x\}=\begin{cases}+1 & \text{if } x>0\\ 0 & \text{if } x=0\\ -1 & \text{if } x<0\end{cases} \qquad (2.11)$$

and $|\,|$ means the expression inside the absolute value signs. For $q=2n$, n is an odd integer, both the sgn$\{x\}$ function and the absolute value sign can be omitted, and expression (2.10) defines real correlation functions. If n is even, there is no need to write absolute value signs, but sgn$\{x\}$ should be kept. In any case, for arbitrary q, this expression still behaves much like a correlation function.

Expression (2.8) defines the exponents for these correlation functions. Note that this expression can also be obtained from the spectrum of cancellation exponents introduced by Du, Tél, and Ott [10]

$$\chi_s^{(q)}(r)=\sum_i\left[\frac{|\int_{C_i}\omega\,dx|}{|\int_V\omega\,dx|}\right]^q\sim r^{-q\kappa+D_q(q-1)} ; \qquad (2.12)$$

cf. (1.12). To see this, we note simply that

$$\langle|\omega(x+r)\omega(x)|^{q/2}\mathrm{sgn}\{\,|\,|\,\}\rangle\sim r^{-D(q-1)}\chi_s^{(q)}(r) . \qquad (2.13)$$

The formula (2.9) is rather general (though some restrictions will be noted in Sec. II B) and provides the possibility for independent measurements of multifractal behavior of turbulence or of general random processes. Indeed the measurements of generalized structure exponent ζ_q by (2.7) would determine the intermittency spectrum D_q and the cancellation exponent κ.

The cancellation exponent can be recovered by setting $q=1$ in the structure or correlation functions. This may be considered to be another definition of κ: one is due to formulas (1.11)–(1.13) and (2.3) and another is with the help of the structure function exponent, with $q=1$. Both definitions give the same κ, because of the simple relation (2.1), as shown by Vainshtein, Du, and Sreenivasan [11]. This is not true for two- or three-dimensional cases, as will be seen in Sec. III.

Once κ is found, any generalized dimension D_q with $q\neq1$ can be determined. In order to find D_1, i.e., information dimension, or entropy, we use the definition

$$\left\langle\frac{|\omega(r)|}{\langle|\omega|\rangle}\ln\frac{|\omega(r)|}{\langle|\omega|\rangle}\right\rangle=\ln\left[\frac{r}{l}\right]^{-(D-D_1)} . \qquad (2.14)$$

We follow the same steps for obtaining (2.2) from (1.7), namely, considering the expression (2.14) at the cutoff scale $r=r_*$ and making use of (1.13) and (2.3) to express $\langle|\omega|\rangle$ through ω_l and the cancellation exponent κ. We get

$$\left\langle\frac{|\omega|}{\omega_l}\ln\frac{|\omega|}{\omega_l}\right\rangle=\left[\frac{r_*}{l}\right]^{-\kappa}\ln\left[\frac{r_*}{l}\right]^{-(\kappa+D-D_1)} . \qquad (2.15)$$

Replacing again $|\omega|$ by $\Delta v(r_*)/r_*$, with the same assumption on scaling that leads to Eqs. (2.7)–(2.9), we obtain, for *the information (or entropy) structure function,* the relation

$$\left\langle\frac{\Delta v}{v_l}\ln\frac{\Delta v}{v_l}\right\rangle=\left[\frac{r}{l}\right]^{1-\kappa}\ln\left[\frac{r}{l}\right]^{[(1-\kappa)-(D-D_1)]} . \qquad (2.16)$$

Note that, in general, the generalized dimensions D_q can be obtained from

$$D-D_q=-\frac{1}{q-1}\lim_{r\to0}\left[q\kappa+\frac{\ln\langle[\Delta v(x+r)/r\omega_l]^q\rangle}{\ln r/l}\right] . \qquad (2.17)$$

If $q\to1$, the expression for $D-D_1$ follows from l'Hôpital's rule. Analogously, the exponent κ follows from

$$\kappa=-\lim_{r\to0}\frac{\ln\langle|\Delta v(x,r)|/r\omega_l\rangle}{\ln r/l} . \qquad (2.18)$$

B. Characteristics of the cancellation exponent and generalized dimensions

As shown by Mandelbrot [5] and Hentschel and Procaccia [6] the generalized dimensions D_q are monotonically decreasing with q. Since $D_q \geq 0$, the function D_q reaches the asymptotic value D_∞ as $q \to \infty$,

$$D_\infty = \min_q \{D_q\} . \tag{2.19}$$

Besides, since all the exponents ζ_q in (2.9) cannot be negative [12], one has

$$0 \leq D - D_q \leq D - D_\infty \leq 1 - \kappa \leq 1 . \tag{2.20}$$

The last inequality (non-negative κ) results in

$$D - D_\infty \leq 1 . \tag{2.21}$$

Since $D = 1$ in this section, this inequality simply means that $D_\infty \geq 0$.

A nontrivial restriction follows from the last inequality in (2.20), namely, $\zeta_1 = 1 - \kappa < 1$. This essentially means that the derivative of the random process v, i.e., random function ω, is singular. If some process v happens not to have singular derivative, then it is possible to obtain such a random function by taking derivatives of that process, if necessary successively until a process is found satisfying the condition $\zeta_1 < 1$.

If v is a one-dimensional cut of a turbulent velocity field, then this singularity condition is satisfied [11]. The velocity field is then stationary (or homogeneous) in the statistical sense. Expression (2.9) is more general and is valid for processes with stationary increments [13], e.g., Brownian motion. For this latter case, $\kappa = \frac{1}{2}$ [14]. The derivative of this process represents white noise, which may have multifractal structure (if $D_q \neq$ const), according to Eq. (2.9). However, the expression for correlation functions (2.10) is *not* valid because white noise (δ-correlated process) has no scaling at all. Thus expression (2.9) for generalized structure functions can be considered rather general, whereas the formula for the correlation function is valid only if the process v is stationary in the statistical sense.

In order to study the asymptotic behavior of generalized dimensions, we write them in the form $D_q = D_\infty + d_q$, where d_q is a positive monotonically decreasing function with increasing q; clearly, $d_q \leq D - D_\infty$. Then,

$$\zeta_q(q) = a + bq + d_q(q - 1) ,$$

where

$$a = D - D_\infty, \quad b = (1 - \kappa) - (D - D_\infty) . \tag{2.22}$$

Asymptotically, as $q \to \infty$,

$$\zeta_q \to bq , \tag{2.23}$$

unless $b = 0$. In the latter case, if $d_q \sim q^{-\alpha}$ as $q \to \infty$, where $0 < \alpha < 1$, then

$$\zeta_q \to q^{1-\alpha} , \tag{2.24}$$

and if d_q decreases faster than q^{-1},

$$\zeta_q \to a . \tag{2.25}$$

Kraichnan [15] suggested that, for scalar fields, $\zeta_q \sim \sqrt{q}$ as $q \to \infty$. This corresponds to the case $b = 0$ and $d(q) \sim q^{-1/2}$.

C. Some examples

(a) Let ζ_q be a linear function of q, $\zeta_q = bq$, $b =$const. Then, by (2.9), $1 - \kappa = b$ and

$$D_q = D . \tag{2.26}$$

There is no intermittency. The situation corresponds to the original scaling theory proposed for turbulence by Kolmogorov [1].

(b) Let $\zeta = a =$const. Then $1 - \kappa = a$ and

$$D_q = D - a = D - 1 + \kappa . \tag{2.27}$$

This corresponds to a monofractal. In the limiting case of $\kappa \to 1$, $D_q = D$ (no intermittency). For this case, $a \to 0$, which corresponds to nondifferentiable function at every point. In fact, if $a = 0$ for an interval $q_0 \leq q \leq q_1$, then $D_q = D$ *for any* q. For this interval,

$$D - D_q = \frac{(1-\kappa)q}{q-1} . \tag{2.28}$$

If $q_0 < 1$, the right-hand side becomes negative, but the left-hand side cannot be negative. If $q_0 \geq 1$, then D_q does not decrease with growing q. The only resolution is to set $\kappa = 1$, and then, by (2.20), $D_q \equiv D$.

In another limiting case, $\kappa \to 0$, $a = 1$, and

$$D_q = D - 1 . \tag{2.29}$$

(c) Consider $\zeta_q = a + bq$. Then

$$1 - \kappa = a + b, \quad D - D_q = a . \tag{2.30}$$

The second equality means that the field is concentrated on a monofractal. If, conversely, we consider a monofractal for which $D_q = D_\beta =$const, then $\zeta_q = a + bq$, where $a = D - D_\beta$, and $b = 1 - \kappa - (D - D_\beta)$.

(d) Consider Burgers's model in the inviscid limit. In this case [16], $\zeta_q = q$ for $0 \leq q \leq 1$ and $\zeta_q = 1$ for $q > 1$. Then $1 - \kappa = 1$, i.e., $\kappa = 0$, and

$$\begin{aligned} D_q &= D \quad \text{for } q < 1 , \\ D_q &= 0 \quad \text{for } q > 1 , \end{aligned} \tag{2.31}$$

so that the structure is *bifractal*.

(e) Consider a scalar field $C(x)$. Suppose that C has a power spectrum of the form $k^{-\beta}$. It follows that

$$\langle |C(x+r) - C(x)|^2 \rangle \sim r^{\beta - 1} . \tag{2.32}$$

Thus $\zeta_2 = \beta - 1$. Since $\zeta_2 \geq 0$, formula (2.32) is valid for $\beta \geq 1$. Using (2.9), we find

$$\zeta_2 = 2(1-\kappa) - (1 - D_2) , \tag{2.33}$$

whence

$$\beta=2(1-\kappa)+D_2 \ . \tag{2.34}$$

Here the cancellation exponent κ is calculated for the signed gradient field $\partial_x C$ and D_2 corresponds to $|\partial_x C|$ field.

From (2.33) we find that there is a tradeoff between the intermittency and cancellation in determining the spectrum. Consider first $D_2=1$, which means that there is no intermittency for the moments with $q\leq 2$. Then the spectral steepness ranges from $\beta=1$ (the Batchelor [17] spectrum for large-eddy dominated advection of the scalar; see also Kraichnan [18]) at $\kappa=1$ to $\beta=3$ for $\kappa=0$. Note that the Batchelor spectrum k^{-1} results in the cancellation exponent $\kappa=1$; according to Eq. (2.20), $D_\infty=1$, i.e., there is no intermittency in the Batchelor regime for all moments, even for asymptotic $q\to\infty$. This was already inferred from the experimental measurements of Ref. [19]. For white noise $\kappa=\frac{1}{2}$ (see Sec. II B) and we recover the Saffman [20] spectrum $\beta=2$. Finally, the Kolmogorov $k^{-5/3}$ spectrum is recovered from (2.34), for $\kappa=\frac{2}{3}$ [11].

Consider now the general case $D_2\leq 1$. According to (2.20), the dimension D_2 cannot be too small $D_2\geq\kappa$ (recall that $D=1$ in this section). Therefore, β ranges from $2-\kappa$ for $D_2=\kappa$ to $3-2\kappa$ for $D_2=1$.

III. TWO- AND THREE-DIMENSIONAL PROCESSES

A. General expressions

Almost all expressions of Sec. II except Eq. (2.1) can be written in vector form. In addition, we may rewrite the first equality in (2.2) (now in vector form) as

$$\langle|\omega|^{q_1q_2}\rangle=\langle|\omega|\rangle^{q_1q_2}\left[\frac{r_*}{l}\right]^{-(D-D_{q_1q_2})(q_1q_2-1)}=\langle|\omega|^{q_1}\rangle^{q_2}\left[\frac{r_*}{l}\right]^{-(D-D_{q_2}^{(q_1)})(q_2-1)}$$
$$=\langle|\omega|\rangle^{q_1q_2}\left[\frac{r_*}{l}\right]^{-(D-D_{q_1})(q_1-1)q_2-(D-D_{q_2}^{(q_1)})(q_2-1)} , \tag{3.1}$$

from which it follows that [21]

$$D_{q_2}^{(q_1)}(q_2-1)=D_{q_1q_2}(q_1q_2-1)-D_{q_1}(q_1-1)q_2 \ . \tag{3.2}$$

Note that the dimensions $D_{q_2}^{(q_1)}$ are based on the measure

$$\mu_\omega^{(q_1)}(C_i)=\frac{\int_{C_i}|\omega|^{q_1}d\mathbf{x}}{\int_V|\omega|^{q_1}d\mathbf{x}} \ . \tag{3.3}$$

In particular, for $q_1=2$,

$$D_q^{(\omega^2)}(q-1)=D_{2q}(2q-1)-qD_2 \ . \tag{3.4}$$

The first consequence of this formula is that

$$D_\infty^{(\omega^2)}=2D_\infty-D_2 \ . \tag{3.5}$$

The second consequence is that if the measure μ_ω corresponds to a monofractal, i.e., $D_q=D_\beta=\text{const}$, then the measure μ_{ω^2} is also monofractal $D_q^{(\omega^2)}=D_\beta$. The converse statement is also true.

The sign-singular measure and therefore the cancellation exponent κ do not make sense for positive quantities because they do not change sign. But correlation functions for them can be expressed through the generalized dimensions. As an example, consider ϵ, the energy dissipation rate. We write for it

$$\langle\epsilon^q\rangle=\langle\epsilon\rangle^q\left[\frac{r_*}{l}\right]^{-(D-D_q^{(\epsilon)})(q-1)} , \tag{3.6}$$

exactly as in (1.7), or rather the first equality in (2.2). Here $D_q^{(\epsilon)}$ is based on the measure

$$\mu_\epsilon(C_i)=\frac{\int_{C_i}\epsilon\,d\mathbf{x}}{\int_V\epsilon\,d\mathbf{x}} \ . \tag{3.7}$$

Suppose that correlation functions have some scaling; that is, for the general three-dimensional case,

$$\langle[\epsilon(\mathbf{x}+\mathbf{r})\epsilon(\mathbf{x})]^{q/2}\rangle=\langle\epsilon^{q/2}\rangle^2\left[\frac{r}{l}\right]^{-y(q)}$$
$$=\langle\epsilon\rangle^q\left[\frac{r_*}{l}\right]^{-(D-D_{q/2}^{(\epsilon)})(q/2-1)2}\times\left[\frac{r}{l}\right]^{-y(q)} . \tag{3.8}$$

In writing the first relation, we took into account that the correlation disappears as the two points separate to distance $\sim l$ so that $\epsilon(\mathbf{x}+\mathbf{r})$ and $\epsilon(\mathbf{x})$ are statistically independent. The exponent $y(q)$ is defined by fitting the above expression for $\langle\epsilon^q\rangle$ to give

$$\langle[\epsilon(\mathbf{x}+\mathbf{r})\epsilon(\mathbf{x})]^{q/2}\rangle=\langle\epsilon\rangle^q\left[\frac{r_*}{r}\right]^{-(D-D_{q/2}^{(\epsilon)})(q/2-1)2}\times\left[\frac{r}{l}\right]^{-(D-D_q^{(\epsilon)})(q-1)} . \tag{3.9}$$

In particular, for the second order correlation function ($q=2$),

$$\langle\epsilon(\mathbf{x}+\mathbf{r})\epsilon(\mathbf{x})\rangle=\langle\epsilon\rangle^2\left[\frac{r}{l}\right]^{-(D-D_2^{(\epsilon)})} , \tag{3.10}$$

as written by Meneveau and Sreenivasan [7].

B. Specifics of two and three dimensions

For two- or three-dimensional measurements, inequality (2.21) imposes restrictions on the applicability of the theory. When the last inequality in (2.20), and therefore the expression (2.21), is violated, the formulas for correlation and structure functions involving cancellation exponent κ, namely, Eqs. (2.8) and (2.9), are not true [11].

Perhaps the most important aspect of $1-D$ cuts of a two- or three-dimensional process is the general formula for intersections [22]

$$D-D_q^{(D)}=1-D_q^{(1)} . \tag{3.11}$$

Here D stands for dimension of space and $D_q^{(1)}$ are the generalized dimensions of one-dimensional intersection. For example, for three-dimensional space,

$$D_q^{(1)}=D_q^{(3)}-2 . \tag{3.12}$$

This formula leads to a paradox. To understand this, suppose that one-dimensional (or filament) structures of the sort described in Ref. [23] are dominant and that the generalized dimensions are close to one (which is not the case in real turbulence), that is to say, that $D_q^{(3)}\gtrsim 1$. According to (3.12), this would lead to negative dimension $D_q^{(1)}$. This happens because one-dimensional cuts do not "notice" structures with dimensions less than two. Since typical dimensions $D_q^{(3)}>2$ in real turbulence [23], the formula (3.12) would not lead to a paradox. However, one-dimensional measurements would underestimate intermittency, or give higher values of $D_q^{(3)}$ than real ones, if calculated according to (3.12). The reason again is the "blindness" of one-dimensional cuts to the structures with dimensions less than two. It is worth noting that with sufficiently long records enough rare events can be captured to allow the detection of negative dimensions even in one-dimensional cuts [24].

Although the formulas of Sec. II are trivially generalized to three dimensions, some of them change their meaning. The most important change concerns the cancellation exponent. In three dimensions, the sum

$$\chi_s(r)=\sum_i \frac{|\int_{C_i}\omega\, d\mathbf{x}|}{|\int_V \omega\, d\mathbf{x}|} \tag{3.13}$$

is constructed [cf. (1.12)] and a scaling relation similar to (1.13) is found. Since we do not have the relationship (2.1), the exponent of the structure function of order one generally does not coincide with the cancellation exponent [11]. In other words, the cancellation exponent defined from scaling (1.13) would differ from κ obtained according to (2.18).

Trivial generalization of the first part of expression (2.2) yields

$$\langle |\omega|^q\rangle = C_q'\langle |\omega|\rangle^q \left[\frac{r_*}{l}\right]^{-(D-D_q)(q-1)} . \tag{3.14}$$

Suppose that, analogous to (2.3), we write

$$\langle |\omega|\rangle=\omega_l\left[\frac{r_*}{l}\right]^{-\kappa'} , \tag{3.15}$$

except that κ' does not necessarily coincide with cancellation exponent κ. Substituting (3.15) into (3.14), we recover the second part of equality (2.2), but with κ' replacing κ. Further denote in three dimensions

$$\Delta v=|\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})| .$$

Since $\Delta v\to|\omega|r_*$ as $r\to r_*$, the quantity $S_q(r)/r^q\sim\langle\Delta v^q\rangle/r^q$ behaves like (2.8) with κ' instead of κ. This allows us to unite the exponents of the generalized structure functions as

$$\zeta_q=(1-\kappa')q-(D-D_q)(q-1) \tag{3.16}$$

instead of (2.9).

C. Two-dimensional measurements: Circulation function

Formula (3.11) for $D=2$ takes the form

$$D_q^{(1)}=D_q^{(2)}-1 . \tag{3.17}$$

It is clear that two-dimensional measurements typically do not notice structures with dimensions less than one. This is much better than one-dimensional cuts and might be quite sufficient for turbulent processes because we do not expect this kind of structure to appear in turbulence.

Let us introduce the circulation function [25]

$$\langle|\Gamma(r)|^q\rangle\sim r^{\alpha_q} , \tag{3.18}$$

where $\Gamma(r)$ is circulation around a two-dimensional box of size r,

$$\Gamma(r)=\oint_{C_i}\mathbf{v}\cdot d\mathbf{s}=\int_{S_i}\omega\cdot d\mathbf{S} ,$$

$\mathbf{S}$ is the vector element of area, and $\mathbf{s}$ is the vector path element. The surface S_i of the box $(=r^2)$ is bounded by the line C_i. Note that absolute value signs can be dropped in (3.18) for even q.

It is clear that

$$\mu_s(C_i)=\frac{\Gamma(r)}{|\Gamma(1)|}$$

would be a simple generalization of the sign-singular measure (1.11) to two-dimensional case and therefore the scaling (1.13) with true cancellation exponent can be recovered.

Now, since $|\Gamma(r_*)|\sim|\omega|r_*^2$, we write, following (3.14),

$$\begin{aligned}\langle|\Gamma(r_*)|^q\rangle&=\tilde{C}_q\langle|\Gamma(r_*)|\rangle^q\left[\frac{r_*}{l}\right]^{-(D-D_q)(q-1)}\\&=\tilde{C}_q\langle|\omega|\rangle^q l^2\left[\frac{r_*}{l}\right]^{2-(D-D_q)(q-1)} .\end{aligned} \tag{3.19}$$

Making use of (2.3) or (1.13), we get, for the exponent of the circulation function (3.18),

$$\alpha_q=(2-\kappa)q-(D-D_q)(q-1) . \tag{3.20}$$

The circulation function (3.18) is in a way more convenient than definition (2.10) because it does not contain the sgn{| |}. The cancellation exponent $\kappa=0.45$ was measured from one-dimensional cuts of the circulation function by Ott *et al.* [9]. As mentioned earlier, these measurements are "blind" to structures with dimensions less than two; not unexpectedly, the κ obtained from circulation function of order one differs from 0.45 [26]. Measurements of the circulation function with $q=1,2,3,4,5,6$ at low or moderate Reynolds numbers of 1100 and 4500, taken from Ref. [26], are shown in Fig. 1. They provide values of κ and generalized dimensions D_2-D_6 (Figs. 2 and 3). These measurements are not only free of restrictions imposed by one-dimensional cuts, but also do not suffer from the limitations of Taylor's hypothesis.

The κ obtained from the circulation function is larger than 0.45, namely, $\kappa \approx 0.85$. We suggest a simple explanation for this fact. Some of the very small vortices cannot be resolved by one-dimensional cuts, which therefore lose information about them. However, these vortices do possess finite circulation as observed on larger scales—as if they are "virtual," that is, they cannot be detected directly, but manifest in finite circulation. Taking these small eddies into account increases the ratio $\langle|\omega|\rangle/\omega_l$, which, according to (2.3), corresponds to increased κ.

We finally note that the traditional way of measuring the multifractal structure of the turbulence velocity increments is based on the refined Kolmogorov hypotheses (see Sec. V B below), namely, by relating these structure functions to the multifractal spectrum of energy dissipation rate ϵ—which is interesting in its own right. As noted in Ref. [23], the nonlinear interaction is weak in filament structures. It has been pointed out in Ref. [16] that this makes it difficult to relate the multifractal structure

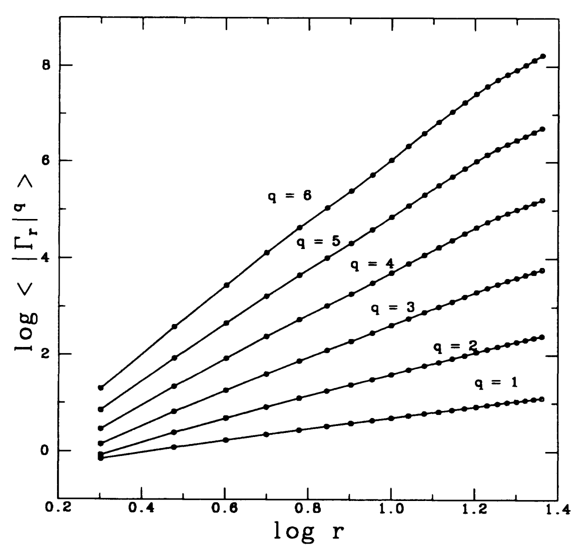

FIG. 1. The circulation functions up to order 6, plotted in $\log_{10}$-$\log_{10}$ coordinates, for the wake of a circular cylinder. The Reynolds number based on the cylinder diameter is 4500 and the boxes around which circulation was measured were centered at 50 diameters from the cylinder. The plane of circulation measurements is that of spatial homogeneity, that is, the plane containing the span of the cylinder and streamwise direction. The velocity field was measured by particle-image velocimetry.

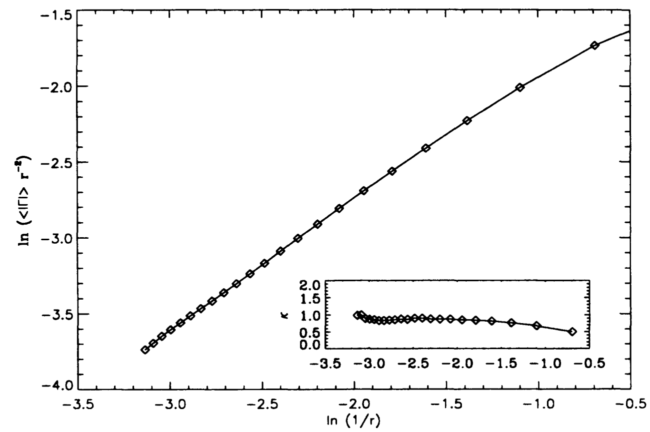

FIG. 2. The cancellation exponent for the circulation function of the first order and the data for the Reynolds number of 4500 shown in Fig. 1.

of ϵ to that of velocity increments (because ϵ actually vanishes in these filaments). The circulation function is free of this difficulty.

IV. CHARACTERISTIC VALUES

Let us recall that typical fields, or characteristic values, have been introduced in the framework of the β model by Frisch, Sulem, and Nelkin [4]. This picture should be modified if the field is concentrated on multifractals. Indeed, suppose that we estimate $\langle|\omega|\rangle \sim \omega_t\beta$, $\langle|\omega|^2\rangle \sim \omega_t^2\beta$, and $\langle|\omega|^4\rangle \sim \omega_t^4\beta$. Here ω_t is typical field and β is the fraction of the volume occupied by the field. Then, from the first two estimates we get

$$\omega_t = \frac{\langle|\omega|^2\rangle}{\langle|\omega|\rangle}, \quad \beta = \frac{\langle|\omega|\rangle^2}{\langle|\omega|^2\rangle} . \tag{4.1}$$

From the first and third estimates we obtain

$$\omega_t = \left[\frac{\langle|\omega|^3\rangle}{\langle|\omega|\rangle}\right]^{1/2}, \quad \beta = \frac{\langle|\omega|\rangle^{3/2}}{\langle|\omega|^3\rangle^{1/2}} . \tag{4.2}$$

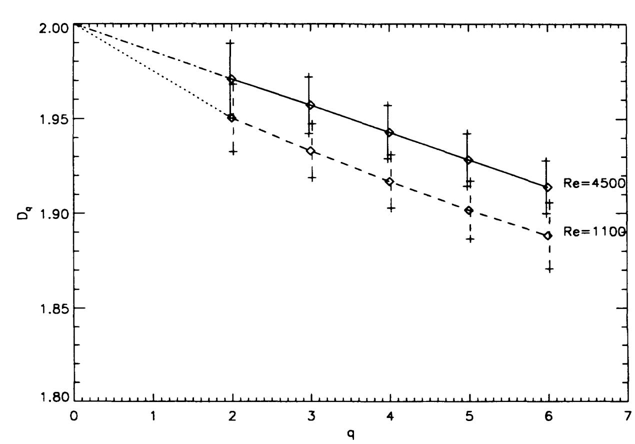

FIG. 3. The generalized dimensions D_2-D_6 for both Reynolds numbers. The significance of the differences between the two Reynolds numbers is not entirely clear.

Then, from the first equality of (2.2) we have

$$\omega_t=\langle|\omega|\rangle\left[\frac{r_*}{l}\right]^{-(D-D_2)}, \quad \beta=\left[\frac{r_*}{l}\right]^{D-D_2}, \tag{4.3}$$

corresponding to (4.1), and

$$\omega_t=\langle|\omega|\rangle\left[\frac{r_*}{l}\right]^{-(D-D_a)}, \quad \beta=\left[\frac{r_*}{l}\right]^{D-D_3}, \tag{4.4}$$

corresponding to (4.2). These two results coincide only if $D_2=D_3=D_\beta$, i.e., *for a monofractal*. A genuinely typical value should, of course, be independent of the index.

In order to take into account of the fact that different moments of the field occupy different volumes, we write

$$\begin{aligned}
&\langle|\omega|\rangle=\omega_t\beta_1 ,\\
&\langle|\omega|^2\rangle=\omega_t^2\beta_2 ,\\
&\langle|\omega|^3\rangle=\omega_t^3\beta_3 ,\\
&\quad\vdots\\
&\langle|\omega|^m\rangle=\omega_t^m\beta_m ,\\
&\quad\vdots\\
&\langle|\omega|^{m+p}\rangle=\omega_t^{m+p}\beta_{m+p} .
\end{aligned} \tag{4.5}$$

This chain of equations can be solved considering high-order moments for which $m \gg 1$. Corresponding generalized dimensions D_m reach their asymptotic value D_∞. Then the typical field

$$\begin{aligned}
\omega_t=\left[\frac{\langle|\omega|^m\rangle}{\langle|\omega|\rangle}\right]^{1/(m-1)}&=\left[\frac{\langle|\omega|^{m+p}\rangle}{\langle|\omega|\rangle}\right]^{1/(m+p-1)}\\
&=\langle|\omega|\rangle\left[\frac{r_*}{l}\right]^{-(D-D_\infty)}\\
&=\omega_l\left[\frac{r_*}{l}\right]^{-[\kappa+D-D_\infty]}
\end{aligned} \tag{4.6}$$

is indeed independent of the index. In the last equality we have again used (2.2). Substituting the typical value ω_t into the lower-order equations of the chain of equations and taking into account (2.2), we get

$$\beta_q=\left[\frac{r_*}{l}\right]^{(D-D_\infty)q-(D-D_q)(q-1)} . \tag{4.7}$$

Note that, in the two limiting cases $q\rightarrow 1$ and $q\rightarrow\infty$,

$$\beta_q\rightarrow\left[\frac{r_*}{l}\right]^{D-D_\infty} . \tag{4.8}$$

An analogous calculation for the chain equations, but written for any $r>r_*$, results in

$$\omega_t(r)=\omega_l\left[\frac{r}{l}\right]^{-[\kappa+D-D_\infty]} \tag{4.9}$$

and

$$\beta_q(r)=\left[\frac{r}{l}\right]^{(D-D_\infty)q-(D-D_q)(q-1)} . \tag{4.10}$$

V. VELOCITY FIELD

A. Relationships involving Kolmogorov's "$\frac{4}{5}$ law"

According to the Kolmogorov "$\frac{4}{5}$ law,"

$$\langle[\Delta v_L(\mathbf{r})]^3\rangle=-\tfrac{4}{5}\langle\epsilon\rangle r+6\nu\partial_r S_2(r) . \tag{5.1}$$

The index L stands for the longitudinal velocity component (i.e., parallel to the vector $\mathbf{r}$) and ν is the kinematic viscosity coefficient. This equation is exact for locally homogeneous and isotropic turbulence. In vector form, (5.1) can be written as

$$\langle[\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})]^2[\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})]\rangle =-\tfrac{4}{3}\langle\epsilon\rangle\mathbf{r}+6\nu\nabla\cdot S_2(r) .$$

For high Reynolds numbers, the viscous term can be neglected, and it therefore follows that

$$\langle[\Delta v_L(r)]^3\rangle\sim r ; \tag{5.2}$$

cf. (1.6). Interestingly, the exponent for the third-order generalized structure function is indistinguishable from that for the structure function $\langle[\Delta v_L(r)]^3\rangle$. That is,

$$\xi_3=\zeta_3=1 . \tag{5.3}$$

There is no formal proof for this observation [3], but it is consistent with Kolmogorov's refined hypothesis [27]; see also Sec. V B. Substitution of (5.3) into (2.9) with $q=1,2,3$ and arbitrary q leads to the elimination of κ and consequently results in

$$\zeta_1=1-\kappa=\tfrac{1}{3}+\tfrac{2}{3}(D-D_3) , \tag{5.4}$$

$$\zeta_2=2(1-\kappa)-(D-D_2)=\tfrac{2}{3}+(D_2-D_3)+\tfrac{1}{3}(D-D_3) , \tag{5.5}$$

$$\zeta_3=3(1-\kappa)-2(D-D_3)=1 , \tag{5.6}$$

and

$$\zeta_q=\frac{q}{3}+(q-1)(D_q-D_3)-\left[\frac{q}{3}-1\right](D-D_3) . \tag{5.7}$$

We see that corrections to Kolmogorov's $\frac{1}{3}$ in (5.4) and $\frac{2}{3}$ in (5.5) are positive since $D_2\geq D_3$ and $D\geq D_3$. Generally, corrections to $q/3$ in (5.7) are positive for $q<3$ and negative for $q>3$. This is consistent with the experimental results of Meneveau and Sreenivasan [7].

Note that for monofractals, $D_q=D_3=D_\beta$, so that

$$\zeta_q=\frac{q}{3}-\left[\frac{q}{3}-1\right](D-D_\beta) . \tag{5.8}$$

For nonintermittent turbulence, $D=D_q$ and

$$\zeta_q=\frac{q}{3} . \tag{5.9}$$

We emphasize that in order to obtain Kolmogorov exponents (5.9) we *did not* use Kolmogorov hypotheses (self-similarity, locality, dimension analysis, etc.) This expression is valid for high-Reynolds-number turbulence as long as (5.3) is true and there is no intermittency. The first demand is not an exact result and the second is definitely not realistic. Still, the exponents $q/3$ are not trivial because, as we saw in example (a) of Sec. II C, any linear function $\zeta_q = bq$ would result in $D_q = D$, i.e., the absence of intermittency. For example, a Gaussian velocity field (no intermittency by definition) results in $\zeta_q = bq$ with arbitrary b.

It is noteworthy that the difference between multifractal and monofractal behavior reveals itself *for small q*. Indeed, from experimental data, $D_q \lesssim D$ for $q<1$. If we substitute $D_q = D$ into (5.7), we get corrections to $q/3 \sim \frac{2}{3}q(D-D_3) \to 0$ as $q \to 0$, whereas for monofractal, by (5.8), the corrections are $\sim(1-\frac{1}{3}q)(D-D_\beta) \to D-D_\beta$ as $q \to 0$.

B. The Kolmogorov hypotheses

If we utilize the refined hypotheses of Kolmogorov

$$S_q(r) = \left[\frac{r}{l}\right]^{q/3} \langle \epsilon^{q/3} \rangle ,$$

where ϵ is energy dissipation rate, we obtain the Meneveau-Sreenivasan formula [28]

$$\zeta_q = \frac{q}{3} - (D - D_{q/3}^{(\epsilon)}) \left[\frac{q}{3} - 1\right] . \tag{5.10}$$

The last term on the right-hand side appears analogous to (1.7) with $q \Longrightarrow q/3$, and generalized dimensions $D_q^{(\epsilon)}$ are based on the measure (3.7). Comparing expression (5.10) with the exact formula (2.9) and (5.7), we get

$$D_{q/3}^{(\epsilon)} = D_3 + \frac{q-1}{q/3-1}(D_q - D_3) , \tag{5.11}$$

so that

$$D_{1/3}^{(\epsilon)} = D_3 .$$

In particular, when $q \to \infty$,

$$D_\infty^{(\epsilon)} = 3D_\infty - 2D_3 . \tag{5.12}$$

Since $\epsilon = \nu\langle \omega^2 \rangle$, $D_q^{(\epsilon)} = D_q^{(\omega^2)}$, and formulas obtained in Sec. III A, provide additional connections between $D_q^{(\epsilon)}$ and D_q. Comparing (3.5) and (5.12), we have

$$D_\infty = 2D_3 - D_2 \tag{5.13}$$

or

$$d_\infty = 2d_3 - d_2 . \tag{5.14}$$

In other words, within framework of Kolmogorov's refined hypotheses, *the knowledge of the intermittency spectrum up to* $q=3$ *makes it possible to define* D_∞ *or* d_∞.

Returning to the characteristic value (4.9), and taking (5.4) into account, we get

$$\omega_t(r) = \omega_l \left[\frac{r}{l}\right]^{-[2/3-(1/3)(D-D_\infty^{(\epsilon)})]} . \tag{5.15}$$

Finally, for $q=2$,

$$\zeta_2 = \tfrac{2}{3} + \tfrac{1}{3}(D - D_{2/3}^{(\epsilon)}) , \tag{5.16}$$

so that

$$\langle \omega(\mathbf{x}+\mathbf{r})\cdot\omega(\mathbf{x}) \rangle = \omega_l^2 \left[\frac{r}{l}\right]^{-[4/3-(1/3)(D-D_{2/3}^{(\epsilon)})]} . \tag{5.17}$$

Therefore,

$$\langle |\omega|^2 \rangle = \omega_l^2 \left[\frac{r_*}{l}\right]^{-[4/3-(1/3)(D-D_{2/3}^{(\epsilon)})]} . \tag{5.18}$$

In the next section, we will use expression (5.18) to estimate the lifetime of the eddies.

VI. LIFETIME OF EDDIES

Since small eddies, having smaller velocities, are advected by the large ones, the correlation time in r space is $\sim r/v_l$. This does not correspond to either the turnover time or the lifetime of small eddies. On the other hand, in Fourier space, the correlation time is essentially the lifetime of the eddy with wave vector k, which we denote $\tau(k)$. We denote the corresponding lifetime of eddies of the size $r=1/k$ as $\tau(r)=\tau(k=1/r)$.

The simplest way to obtain the lifetime is to invoke the Kolmogorov hypothesis on energy flux conservation

$$\frac{S_2(r)}{\tau(r)} = \text{const} , \tag{6.1}$$

from which, using (5.16), it follows that

$$\tau(r) = \frac{l}{\langle |\mathbf{v}|^2 \rangle^{1/2}} \left[\frac{r}{l}\right]^{2/3-(1/3)(D-D_{2/3}^{(\epsilon)})} . \tag{6.2}$$

The same power law can be obtained from quite different considerations. It follows from energy conservation that

$$\partial_t \langle |\mathbf{v}|^2 \rangle = -\nu \langle |\omega|^2 \rangle = -\langle \epsilon \rangle , \tag{6.3}$$

where, from experimental data [29], $\langle \epsilon \rangle = \langle |\mathbf{v}|^2 \rangle^{3/2}/l$. Equation (6.3) then implies that

$$\partial_t \langle |\mathbf{v}|^2 \rangle = -\frac{\langle |\mathbf{v}|^2 \rangle^{3/2}}{l} \approx -\frac{\langle |\mathbf{v}|^2 \rangle}{\tau(l)} , \tag{6.4}$$

where

$$\tau(l) = \frac{l}{\langle |\mathbf{v}|^2 \rangle^{1/2}} . \tag{6.5}$$

On the other hand, substitution of (5.18) into (6.3) defines

$$\frac{r_*}{l} = \mathrm{Re}^{-1/[4/3-(1/3)(D-D_{2/3}^{(\epsilon)})]} . \tag{6.6}$$

Here the Reynolds number Re is defined by $\langle |\mathbf{v}|^2 \rangle^{1/2} l/\nu$.

Note that the lifetime of the eddy cannot exceed the viscous time r^2/ν. In the inertial range where viscosity is negligible, $\tau(r) \gg r^2/\nu$. As we will see, $\tau(r)$ is also much larger than the turnover time of eddies of size r. Therefore at $r=r_*$

$$\tau(r_*)=\frac{r_*^2}{\nu} . \tag{6.7}$$

Now the only power law for $\tau(r)$ that fits both (6.5) and (6.7) corresponds to (6.2).

Note that the β model of Frisch, Sulem, and Nelkin [4] also gives (6.2) and $D_{2/3}^{(\epsilon)}=D_\beta=\text{const}$ (monofractal).

The lifetime of the eddies given by (6.2) can be compared with the turnover time. The latter is simply defined as a characteristic value $\tau_t(r)$,

$$\tau_t(r)=\frac{1}{\omega_t(r)} , \tag{6.8}$$

where $\omega_t(r)$ is defined in (5.15). Thus

$$\tau_t(r)=\frac{l}{\langle |\mathbf{v}|^2\rangle^{1/2}}\left[\frac{r}{l}\right]^{2/3-(1/3)(D-D_\infty^{(\epsilon)})} , \tag{6.9}$$

which is *much shorter* than the lifetime

$$\frac{\tau_t(r)}{\tau(r)}=\left[\frac{r}{l}\right]^{(1/3)(D_{2/3}^{(\epsilon)}-D_\infty^{(\epsilon)})} \ll 1 . \tag{6.10}$$

Indeed $D_{2/3}^{(\epsilon)}\geq D_\infty^{(\epsilon)}$.

It is interesting to note that this persistence of eddies can be explained only in the multifractal model because, for monofractal, $D_{2/3}^{(\epsilon)}=D_\infty^{(\epsilon)}=D_\beta$.

Measurements of D_q at large q are quite difficult and, to our knowledge, have not been made. (They have been estimated from the measurements of $D_q^{(\epsilon)}$; see [28].) However, the lifetimes of various scales of passive dyes scalars have been obtained directly for turbulent jets using wavelet representation. These estimates [30] show extraordinary persistence of small scales of the dye field. This extraordinary persistence of eddies is in itself an indication of the multifractal structure of turbulence. On the other hand, measurements of correlation time $\tau(r)$ and turnover time of the eddies might give an estimation of the quantity D_∞.

VII. PROBABILITY DISTRIBUTION FUNCTION OF VELOCITY INCREMENTS

A. General considerations

If the scaling experiments ζ_q and the prefactors of all moments $\tilde{S}_q$ of the quantity $\Delta v=|\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})|$ are known, the probability distribution function $P(\Delta v,r)$ is unambiguously defined. Thus we can write

$$\int P(\Delta v,r)\Delta v^q d\Delta v=\tilde{S}_q(r) . \tag{7.1}$$

Performing the change of variables

$$\Delta v=cr^h\left[=v_l\left[\frac{r}{l}\right]^h\right] , \tag{7.2}$$

we may write (7.1) as

$$\int \tilde{P}(h,r)c^q r^{qh}dh=\tilde{S}_q(r) , \tag{7.3}$$

where

$$\tilde{P}(h,r)=P(cr^h,r)cr^h\ln r . \tag{7.4}$$

To obtain $\tilde{P}(h,r)$ from (7.3) for given $\tilde{S}_a(r)$, we can apply the inverse Legendre transformation to this equation. Then $\tilde{P}(h,r)$ is found to be of the form

$$\tilde{P}(h,r)=\rho(h)|\ln r|^{1/2}r^{\tilde{f}(h)} . \tag{7.5}$$

The appearance of a weak dependence on r in the logarithmic factor will be clarified later. Now, working backward by substituting $\tilde{P}(h,r)$ from (7.5) into (7.3), we get the Legendre transform. As a result, one has

$$\zeta_q=\min_h\{D-\tilde{D}(h)+qh\} , \tag{7.6}$$

$$D-\tilde{D}(h)=\tilde{f}(h) . \tag{7.7}$$

Here $\tilde{D}(h)$ is the Hausdorff dimension of the subset of singularities of the strength h, as in (7.2); see Frisch and Parisi [6]. As already mentioned in Sec. II, ζ_q cannot be negative. This follows from (7.6), namely, ζ_q is a minimum for some non-negative value of the expression within and cannot be negative. Indeed, $D-\tilde{D}(h)$ is non-negative and so is qh.

Returning to the inverse Legendre transform, we have

$$\tilde{D}(h)=\min_q\{D-\zeta_q+qh\} . \tag{7.8}$$

If ζ_q is known from experiment, then (7.6) can be used to define the Hausdorff dimension $\tilde{D}(h)$; this equality is equivalent to steepest descent calculation. Then the integral in (7.3) is estimated as

$$\rho(h_1)|\ln r|^{1/2}\left[\frac{2\pi}{|\tilde{D}''(h_1)\ln r|}\right]^{1/2}r^{D-\tilde{D}(h_1)+qh_1}$$
$$=\rho(h_1)\left[\frac{2\pi}{|\tilde{D}''(h_1)|}\right]^{1/2}r^{D-\tilde{D}(h_1)+qh_1} , \tag{7.9}$$

where h_1 is the root of equation

$$\frac{\partial\tilde{D}}{\partial h}=q \tag{7.10}$$

and $\tilde{D}''(h_1)$ should satisfy

$$\tilde{D}''(h_1)=\left.\frac{\partial^2\tilde{D}}{\partial h^2}\right|_{h=h_1}<0 . \tag{7.11}$$

It is now clear from (7.9) that the logarithmic factor in (7.5) was introduced in order to cancel that in the final expression (7.9).

Equating (7.9) with the right-hand side of (7.3), we obtain

$$\tilde{D}(h_1)=D-\zeta_q+qh_1 , \tag{7.12}$$

which should be compared with (7.8) and

$$\rho(h_1)=\left[\frac{|\tilde{D}''(h_1)|}{2\pi}\right]^{1/2} . \tag{7.13}$$

Now h_1 is a function of q as can be seen from (7.10). But, instead of using this equation, one usually differentiates

(7.12) with respect to q to obtain

$$h_1(q)=\frac{d}{dq}\zeta_q \ . \tag{7.14}$$

The solution of Eq. (7.14) for q as a function of h_1, i.e., $q(h_1)$, is substituted into (7.12) and (7.13) to give the Hausdorff dimension $\tilde{D}(h)$ and $\rho(h)$. Finally, $P(\Delta v,r)$ can be recovered backward, using (7.5) and (7.4).

B. Characteristics of the probability density function

Consider a self-similar PDF

$$P(\Delta v,r)\Longrightarrow \left[\frac{r}{l}\right]^{-b}\hat{P}(y), \quad y=\frac{\Delta v}{v_l}\left[\frac{r}{l}\right]^{-b} . \tag{7.15}$$

It is easy to see that this PDF results in $\tilde{S}_q(r)\sim r^{bq}$. According to example (a) of Sec. II C, this corresponds to nonintermittent random process $D_q=D$ and $b=1-\kappa$. The converse statement is also true. Indeed, Eq. (7.1) is the Mellin transform. Applying the inversion integral to $S_q(r)\sim r^{bq}$, we recover (7.15). Recall that, according to example (a), this case corresponds to no intermittency. Note that $\hat{P}(y)$ is essentially an arbitrary function, not necessarily Gaussian.

For $q=0$, the left-hand side of (7.1) should give

$$\int P(\Delta v,r)d\Delta v=1 \ . \tag{7.16}$$

But, the right-hand side of (7.1) is

$$S_0(r)=\left[\frac{r}{l}\right]^{D-D_0} , \tag{7.17}$$

which is less than 1 if

$$D-D_0>0 \ . \tag{7.18}$$

The last inequality, that the Hausdorff dimension is less than the space dimension, simply means that the zero value field appears with finite probability. Or, in explicit form,

$$P(\Delta v,r)=\left[1-\left[\frac{r}{l}\right]^{D-D_0}\right]\delta(\Delta v)+P_1(\Delta v,r) \ . \tag{7.19}$$

In order to satisfy (7.16),

$$\int P_1(\Delta v,r)d\Delta v=\left[\frac{r}{l}\right]^{D-D_0} . \tag{7.20}$$

This PDF does give the right answer for (7.1); this is so because, if $q\neq 0$, however small, the first term on the right-hand side of (7.19) does not make any contribution.

It is important to note that inequality (7.18) is definitely satisfied for monofractals because $D_q=D_\beta=D_0<D$. Therefore, if turbulent processes are concentrated on monofractals, the δ function should be observed at the origin, as in (7.19). Of course, in real measurements, instead of the δ function, one will find a peak at $\Delta v=0$, with a width determined by the noise of errors. Such a peak is usually not observed. Instead, usually $D_0=D$ and $D_q\lesssim D$ for $q<1$. Another feature of monofractals is that $P_1(\Delta v,r)$ appears in self-similar form (7.15), with $b=(1-\kappa)-(D-D_\beta)$. Experimental data [31] show that the PDF is of the form

$$P(\Delta v)\sim \exp\{-y^{m(r)}\} , \tag{7.21}$$

where y is defined by (7.15), $b=\frac{1}{3}$, and $m(r)$ is *not* a constant in inertial region. The PDF could be self-similar only if $m=$const. The fact that m is not a constant is an additional indication of multifractal nature of turbulence.

Perhaps the simplest example of multifractals distribution function is

$$P(\Delta v,r)\sim \hat{P}_1(y_1)+\hat{P}_2(y_2) , \tag{7.22}$$

$$y_1=\frac{\Delta v}{v_l}\left[\frac{r}{l}\right]^{-b_1}, \quad y_2=\frac{\Delta v}{v_l}\left[\frac{r}{l}\right]^{-b_2} , \tag{7.23}$$

where functions $\hat{P}_1$ and $\hat{P}_2$ have different characteristic scales and substantially different amplitudes.

In fact, one may say that *any* PDF would give multifractal structure, unless it can be presented in a self-similar form (7.15), provided, of course, there is a scaling region for the structure functions. However, in order to have pronounced (multifractal) intermittency, the PDF should look like that depicted in Fig. 4.

The multifractality can be characterized by ΔD_q,

$$\Delta D_q=D_0-D_\infty=(D-D_\infty)-(D-D_0)=d_\infty-d_0 \ . \tag{7.24}$$

Usually, one has $D_0=D$, which then gives

$$\Delta D_q=d_\infty \ . \tag{7.25}$$

Thus, if $D_0=D$ and the turbulence is intermittent, i.e, $d_\infty>0$, then, according to (7.25), the process is also multifractal.

In particular, for the velocity field obeying Kolmogorov's refined by hypotheses, the multifractality, according to (5.14), is as follows

$$\Delta D_q=3d_3-d_2 \ . \tag{7.26}$$

However, according to (2.20), the multifractality is constrained in such a way that

$$\Delta D_q\leq 1-\kappa \ . \tag{7.27}$$

Thus measuring the structure function of order 1, we ob-

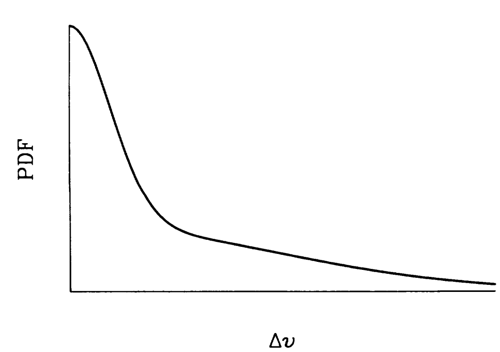

FIG. 4. Sketch of the PDF with pronounced multifractality.

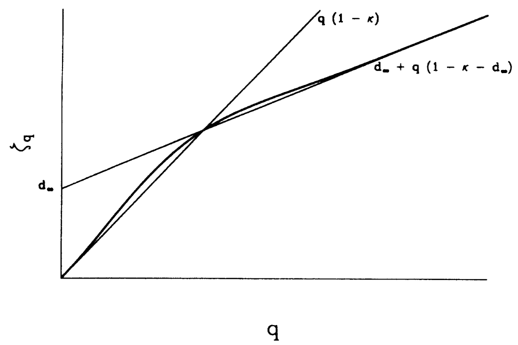

FIG. 5. Typical ζ_q (bold). The straight line $\zeta_q = q(1-\kappa)$ corresponds to the nonintermittent case. The line $\zeta_q = d_\infty + q[(1-\kappa) - d_\infty]$ gives the asymptote of the curve and also corresponds to a monofractal.

tain the quantity $1-\kappa$ and therefore the constraint on multifractality (and actually on the intermittency).

The typical form of ζ_q is sketched in Fig. 5. Here ζ_q grows monotonically with q. Thus, by (7.14), h changes in the interval

$$[(1-\kappa)-d_\infty] \le h \le 1-\kappa \ . \tag{7.28}$$

Again, this interval, being essentially characteristic of multifractality, is maximal if (7.27) is satisfied as an equality.

Let us return now to the nonintermittent case, $D_q = D$. The Hausdorff dimension is defined by (7.8) and is simply $\tilde{D}(h)=0$ for $h < 1-\kappa$ and $\tilde{D}(h)=D$ for $h \ge 1-\kappa$; see Benzi *et al.* [4]. This is not a smooth function and Eqs. (7.9)–(7.14) do not apply. It is clear, however, that if we define PDF $\rho(h)$ as

$$\int \rho(h) r^{D-\tilde{D}(h)} c^q r^{qh} dh = \tilde{S}_q(r) \ , \tag{7.29}$$

then, for the nonintermittent case, $\tilde{D}(h)=D$ and

$$\rho(h) = \delta(h - [1-\kappa]) \ . \tag{7.30}$$

For monofractals, $\tilde{D}(h)=0$ for $h < (1-\kappa)-(D-D_\beta) = \frac{1}{3} - (D-D_\beta)/3$, the last equality being valid for velocity fields [see (5.8)] and $\tilde{D}(h)=D_\beta$ for $h \ge (1-\kappa) - (D-D_\beta)$ (see Benzi *et al.* [4]). As to the PDF, it cannot be presented as in Eq. (7.29). According to (7.19), monofractals incorporate $\delta(\Delta v)$ and finite probability of zeroth velocity would result in $h \to \infty$ in transformation (7.2). Thus only the second term on the right-hand side of (7.19) can be transformed to the PDF $\rho(h)$ as in (7.29) to give

$$\rho(h) = \delta(h - [1-\kappa] + [D - D_\beta]) \ . \tag{7.31}$$

VIII. PRINCIPAL CONCLUSIONS

In the past, various scaling exponents have been defined to quantify intermittency or multifractality of a random process. In this paper we have considered the relationships among several classes of these exponents. A complete analysis is possible for one-dimensional processes or one-dimensional intersections of a three-dimensional process; for these cases, the relation among the scaling exponents for generalized structure functions, generalized dimensions of velocity increments in the inertial range, and the cancellation exponents of the derivative of the process can be stated succinctly in the form of Eq. (2.9). The implications of this relation have been discussed by providing several examples. For two- and three-dimensional processes, certain restrictions apply; they are considered in some detail and the analysis is carried out as far as possible. Of special interest in two dimensions are the so-called circulation functions, which are moments of the absolute values of the circulation around boxes of a given scale. The scaling exponents for circulation functions are again shown to be related to generalized dimensions and the cancellation index through Eq. (3.20).

This material forms the subject of the first three sections of the paper. The relations derived in these sections and the discussions surrounding them are valid for general stochastic processes. While it is true that their internal consistency is established by taking recourse to available turbulence measurements, chiefly its one- and two-dimensional intersections, they are not special to turbulence.

In contrast, the rest of the paper deals primarily with turbulence. Among other things, we have related the qth-order characteristic value [defined through Eq. (4.10)] to generalized dimensions. We have shown under plausible assumptions supported by experiment that the Kolmogorov structure function exponents can be obtained from the $\frac{4}{5}$th law. We have also shown that, within the framework of Kolmogorov's refined hypotheses, the knowledge of the intermittency spectrum up to the third order would be adequate to define D_∞. It is argued in some detail that the extraordinary persistence of eddies of various scales is a strong indication of the multifractality of turbulence structure. It is demonstrated that the probability density function of velocity increments for a monofractal process possesses a similarity form of the stretched exponential type $\exp(-y^m)$ in the similarity variable y defined by Eq. (7.15) and a constant m independent of r. Experiments, on the other hand, show that m for high-Reynolds-number turbulence is a smooth function of r, varying between about $\frac{1}{2}$ for dissipative scales and about 2 for large scales. It is argued that this is a confirmation of multifractality of turbulence.

In summary, it appears that there is overwhelming support for the notion that high-Reynolds-number turbulence possesses many aspects of multifractality. Indeed it would be difficult to reconcile and understand many empirical observations without invoking multifractality.

Note added in proof. Expression (3.10) is usually written in the form

$$\langle \epsilon(\mathbf{x}+\mathbf{r})\epsilon(\mathbf{x}) \rangle = \langle \epsilon^2 \rangle \left[\frac{r}{r_*} \right]^{-\mu}$$

and has been confirmed experimentally by several authors. For a summary of experiments, see Ref. [32].

Note that the confirmation was obtained by Meneveau and Sreenivasan [7] making use of formula $\langle \epsilon(\mathbf{x}+\mathbf{r})\epsilon(\mathbf{x}) \rangle \sim \langle [\epsilon(r)]^2 \rangle$ [see (1.7) and (3.6)], derived by Novikov [33]. The vector form of the Kolmogorov law given in Sec. V A can be obtained by contraction of the tensor expression of Novikov [34].

ACKNOWLEDGMENTS

We thank Roberto Benzi, Robert Kraichnan, Norman Lebovitz, Benoit Mandelbrot, Evgeny Novikov, and Steven Orszag for their interest in the work and valuable comments. The work was supported partially by an AFOSR grant to Yale.

[1] A. N. Kolmogorov, C.R. Acad. Sci. U.S.S.R. **30**, 301 (1941).
[2] A. N. Kolmogorov, C.R. Acad. Sci. U.S.S.R. **32**, 16 (1941).
[3] For example, F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984); R. Benzi, L. Biferale, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli, and S. Succi, Phys. Rev. E **48**, R30 (1993); G. Stolovitzky, K. R. Sreenivasan, and A. Juneja, *ibid.* **48**, R3217 (1993); L. Zubair, Ph.D. thesis, Yale University, 1993.
[4] For example, A. N. Kolmogorov, J. Fluid Mech. **13**, 80 (1962); E. A. Novikov and R. W. Stewart, Izv. Akad. Nauk. SSSR Geofiz. **3**, 408 (1964); E. A. Novikov, Prikl. Mat. Mech. **35**, 266 (1971); U. Frisch, P.-L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978); R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984); C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987); D. Schertzer and S. Lovejoy, in *Turbulent Shear Flows,* edited by L. J. S. Bradbury, F. Durst, B. Launder, F. W. Schmidt, and J. H. Whitelaw (Springer, New York, 1985), pp. 7–33; K. R. Sreenivasan and G. Stolovitsky, J. Stat. Phys. (to be published).
[5] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[6] H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983); U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics,* edited by M. Gil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), pp. 84–88; G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).
[7] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991); R. R. Prasad, C. Meneveau, and K. R. Sreenivasan, Phys. Rev. Lett. **61**, 74 (1988); C. Meneveau, K. R. Sreenivasan, P. Kailasnath, and M. S. Fan, Phys. Rev. A **40**, 894 (1989); E. Aurell, U. Frisch, J. Lutsko, and M. Vergassola, J. Fluid Mech. **238**, 467 (1992); M. Borgas, Philos. Trans. R. Soc. London **342**, 379 (1993).
[8] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[9] E. Ott, Y. Du, K. R. Sreenivasan, A. Juneja, and A. K. Suri, Phys. Rev. Lett. **69**, 2654 (1992); Y. Du and E. Ott, Physica D **67**, 387 (1993).
[10] Y. Du, T. Tél, and E. Ott, Physica D (to be published).
[11] S. I. Vainshtein, Y. Du, and K. R. Sreenivasan, Phys. Rev. E **49**, R2521 (1994).
[12] U. Frisch, Proc. R. Soc. London Ser. A **434**, 89 (1991).
[13] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1971), Vol. 2.
[14] A. Bertozzi and A. B. Chhabra, Phys. Rev. E **49**, 4716 (1994).
[15] R. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[16] E. Aurell, U. Frisch, J. Lutsko, and M. Vergassola, J. Fluid Mech. **238**, 467 (1992).
[17] G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959).
[18] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[19] K. R. Sreenivasan and R. R. Prasad, Physica D **38**, 322 (1989).
[20] P. G. Saffman, Stud. Appl. Math. **50**, 377 (1971).
[21] Y. Du, Ph.D. thesis, University of Maryland, College Park, 1993.
[22] B. B. Mandelbrot, *Fractal Geometry of Nature* (Freeman, New York, 1983); K. R. Sreenivasan, Annu. Rev. Fluid Mech. **23**, 539 (1991).
[23] Z.-S. She, E. Jackson, and S. A. Orszag, Nature **344**, 226 (1990).
[24] A. B. Chhabra and K. R. Sreenivasan, Phys. Rev. Lett. **68**, 2762 (1992).
[25] K. R. Sreenivasan and A. Juneja (unpublished).
[26] A. Juneja, K. R. Sreenivasan, G. Stolovitsky, and A. K. Suri (unpublished).
[27] A. N. Kolmogorov, J. Fluid Mech. **13**, 80 (1962); G. Stolovitzky, P. Kailasnath, and K. R. Sreenivasan, Phys. Rev. Lett. **69**, 1178 (1992); A. Praskovsky, Phys. Fluids A **4**, 2589 (1992); S. T. Thoroddsen and C. W. Van Atta, *ibid.* **4**, 2592 (1992); S. Chen, G. D. Doolen, R. H. Kraichnan, and Z.-S. She, *ibid.* **5**, 458 (1993); I. Hosokawa, J. Phys. Soc. Jpn. **62**, 10 (1993); S. Grossmann and D. Lohse, Phys. Fluids A **6**, 611 (1994); G. Stolovitzky and K. R. Sreenivasan, Rev. Mod. Phys. **66**, 229 (1994).
[28] C. Meneveau and K. R. Sreenivasan, Nucl. Phys. B (Proc. Suppl.) **2**, 49 (1987).
[29] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, England, 1953), Chap. 6; K. R. Sreenivasan, Phys. Fluids **27**, 1048 (1984).
[30] K. R. Sreenivasan and R. M. Everson (unpublished).
[31] P. Kailasnath, K. R. Sreenivasan, and G. Stolovitzky, Phys. Rev. Lett. **68**, 2776 (1992).
[32] K. R. Sreenivasan and P. Kailasnath, Phys. Fluids A **5**, 512 (1993).
[33] E. A. Novikov, Prikl. Math. Mech. **35**, 266 (1971).
[34] E. A. Novikov, Phys. Rev. A **49**, R6147 (1992).